THE REPRESENTATION TRANSFORMATION OF MULTIQUARK WAVE FUNCTIONS

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ABSTRACT

It is shown that the representation transformations of multiquark wave functions between different coupling schemes are just the Racah coefficients of the permutation group. The transformation coefficients between the flavor-spin (FS) and the color-spin (CS) schemes are obtained. As an example, the expansion of the physical bases in terms of symmetry bases in the CS scheme are given for the interesting cases $YIJ = 201, 210, 000$.

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I. Introduction

To understand baryon-baryon interactions, and in turn to search for dibaryon candidates from the fundamental strong interaction theory QCD, is still a challenge to contemporary physics. Due to the complexity of confinement, one relies mainly on QCD inspired models. The central problem in the computations is the many-body matrix element calculation. If one wants to do systematic studies, a powerful method is indispensable.

In the calculation of baryon-baryon interactions and dibaryon masses, of the bases that can be used to span the model Hilbert space[1,2], two are of particular interest. The first is the so-called physical basis, the other, the symmetry basis.

The physical basis is nothing more than the quark cluster model basis, which is constructed directly from two single baryons:

$$\begin{aligned}\Psi_{\alpha\kappa}(B_1B_2) &= \mathcal{A}\left[\psi(B_1)\psi(B_2)\right]^{[\sigma]IJ}_{WM_IM_J} \\ &= \mathcal{A}\left[\left| \begin{array}{c} [\nu_1] \\ [\sigma_1][\mu_1][f_1]Y_1I_1J_1 \end{array} \right\rangle \left| \begin{array}{c} [\nu_2] \\ [\sigma_2][\mu_2][f_2]Y_2I_2J_2 \end{array} \right\rangle\right]^{[\sigma]IJ}_{WM_IM_J}, \end{aligned} \tag{1}$$

where $\alpha = (YIJ)$, κ represents all the other quantum numbers and $\mathcal{A}$ is the pairwise quark antisymmetrization operator. The physical basis has the advantage that it has definite dibaryon content in the asymptotic region, but it has the disadvantage that it is not convenient for the calculation of matrix elements. In order to take advantage of the fractional parentage expansion technique, which has been widely used in atomic and nuclear physics, one has to use the symmetry basis.

The symmetry basis is the group chain classification basis:

$$\Phi_{\alpha K}(q^6) = \left| [\nu] l^3 r^3 \right\rangle. \tag{2}$$

where K represents other quantum numbers and the group chain we use is

$$\begin{aligned} &\mathrm{SU}(36) \supset \mathrm{SU}^x(2) \times (\mathrm{SU}(18) \supset \mathrm{SU}^c(3) \times (\mathrm{SU}(6) \\ &\quad \supset (\mathrm{SU}^f(3) \supset \mathrm{SU}^\tau(2) \times \mathrm{U}^Y(1)) \times \mathrm{SU}^\sigma))(2) \end{aligned} \tag{3}$$

The symmetry basis has its own disadvantage: Most of the quantum numbers in Eq.(3) are just mathematical labels with no physical meaning. Thus they do not describe either conserved quantum numbers, nor definite dibaryon content in the asymptotic region.

To take both the advantages of these two bases, one must first express the physical basis in terms of symmetry basis, then calculate the many-body matrix element using the symmetry basis, and finally transform the results back to the physical basis. Hence, the transformation between the physical and symmetry bases is needed. Since the two bases both are complete in the truncated model Hilbert space, they are related by a unitary transformation. The needed transformation coefficients in the u, d and s 3-flavor world have been tabulated for the flavor-spin (FS) scheme[1,3] specified by Eq.(3).

In different applications, the coupling orders are also different. In the FS scheme, the coupling order is as follows:

(1) $SU^{f\sigma}(6) \supset SU^f(3) \times SU^\sigma(2), \qquad [f] \times [\sigma_J] \to [\mu].$

(2) $SU^{cf\sigma}(18) \supset SU^c(3) \times SU^{f\sigma}(6), \qquad [\sigma] \times [\mu] \to [\tilde{\nu}].$

(3) $SU^{xcf\sigma}(36) \supset SU^x(2) \times SU^{cf\sigma}(18), \qquad [\nu] \times [\tilde{\nu}] \to [1^6].$

In the most popular color-spin (CS) scheme, the coupling order is:

(1) $SU^{c\sigma}(6) \supset SU^c(3) \times SU^\sigma(2), \qquad [\sigma] \times [\sigma_J] \to [\mu'].$

(2) $SU^{c\sigma f}(18) \supset SU^{c\sigma}(6) \times SU^{f}(3), \qquad [\mu'] \times [f] \to [\tilde{\nu}].$

(3) $SU^{xc\sigma f}(36) \supset SU^{x}(2) \times SU^{c\sigma f}(18), \qquad [\nu] \times [\tilde{\nu}] \to [1^6].$

Each of these schemes has its advantages. For example, in the CS scheme, the expectation values of the color magnetic operator

$$\sum_{i<j} \lambda_{\mathbf{i}} \cdot \lambda_{\mathbf{j}} \sigma_{\mathbf{i}} \cdot \sigma_{\mathbf{j}}$$

can be easily calculated. In the FS scheme useful results already exist.

To understand the relation between results obtained in different coupling schemes and make our Tables[3] more useful (for example, to expand the physical bases in terms of symmetry bases in the CS scheme), it is necessary to know the unitary representation (*rep*) transformation between the bases in the different schemes. Fl. Stancu[4] has studied the *rep* transformation in the u, d 2-flavor world and obtained the transformation coefficients for the cases $(YIJ) = (201)$ and (200) as well as several trivial cases.

Because of the plethora of phase conventions possible for the Clebsch-Gordon (CG) coefficients of SU(n) for $n \geq 3$, it is highly desirable to have a systematic and phase consistent method to calculate the *rep* transformation coefficients. In this work, we will prove that the *rep* transformation coefficients are just the Racah coefficients of permutation group, $S(n)$. (See Sec. II). Since the S(n) $\supset$ S($n-1$) isoscalar factors are available[5], the Racah coefficients of S(n) for $n \leq 6$ can be calculated based on the genealogical method. As an application, the expansion of the physical bases in terms of symmetry bases in the CS scheme are obtained. (See Sec. III).

II. Rep transformation coefficients and Racah coefficients of S(n)

It is well known that the irreducible basis (IRB) of SU(mn) $\supset$ SU(m) $\times$ SU(n) can be constructed from the IRBs of SU(m) and SU(n) by using the CG coefficients of permutation group.

$$\left| \begin{array}{c} [\nu] \\ m, \beta[\nu_1]W_1[\nu_1]W_2 \end{array} \right\rangle = \sum_{m_1,m_2} C^{\nu_\beta m}_{\nu_1 m_1,\nu_2 m_2} \left| \begin{array}{c} [\nu_1] \\ m_1 W_1 \end{array} \right\rangle \left| \begin{array}{c} [\nu_2] \\ m_2 W_2 \end{array} \right\rangle . \tag{4}$$

where β is the multiplicity index in the coupling $[\nu_1] \times [\nu_2] \to [\nu]$, m_1, m_2 and m are Yamanouchi labels.

Based on Eq.(4), we can write down multiquark wave functions (for 6-quark color singlets) in the different schemes. The steps to achieve this are:

1. FS scheme:

(1) $[f] \times [\sigma_J] \to [\mu]$

$$\left| \begin{array}{c} [\mu] \\ m_\mu, \beta[f]W_f[\sigma_J]W_J \end{array} \right\rangle = \sum_{m_f,m_J} C^{\mu_\beta m_\mu}_{f m_f,\sigma_J m_J} \left| \begin{array}{c} [f] \\ m_f W_f \end{array} \right\rangle \left| \begin{array}{c} [\sigma_J] \\ m_J W_J \end{array} \right\rangle ; \tag{5}$$

(2) $[\sigma] \times [\mu] \to [\tilde{\nu}]$

$$\left| \begin{array}{c} [\tilde{\nu}] \\ m_{\tilde{\nu}}, \alpha[\mu]\beta[f]W_f[\sigma_J]W_J[\sigma]W_c \end{array} \right\rangle = \sum_{m_f,m_J,m_c,m_\mu} C^{\tilde{\nu}_\alpha m_{\tilde{\nu}}}_{\sigma m_c,\mu m_\mu} C^{\mu_\beta m_\mu}_{f m_f,\sigma_J m_J} \left| \begin{array}{c} [\sigma] \\ m_c W_c \end{array} \right\rangle \left| \begin{array}{c} [f] \\ m_f W_f \end{array} \right\rangle \left| \begin{array}{c} [\sigma_J] \\ m_J W_J \end{array} \right\rangle ; \tag{6}$$

(3) $[\nu] \times [\tilde{\nu}] \to [1^6]$

$$\Phi^{FS} = \sum_{\text{all } m} C^{[1^6]1}_{\nu m_\nu,\tilde{\nu} m_{\tilde{\nu}}} C^{\tilde{\nu}_\alpha m_{\tilde{\nu}}}_{\sigma m_c,\mu m_\mu} C^{\mu_\beta m_\mu}_{f m_f,\sigma_J m_J} \left| \begin{array}{c} [\nu] \\ m_\nu W_\nu \end{array} \right\rangle \left| \begin{array}{c} [\sigma] \\ m_c W_c \end{array} \right\rangle \left| \begin{array}{c} [f] \\ m_f W_f \end{array} \right\rangle \left| \begin{array}{c} [\sigma_J] \\ m_J W_J \end{array} \right\rangle . \tag{7}$$

2. CS scheme:

Following the same steps, the multiquark wave function can be written as

$$\Phi^{CS}=\sum_{\text{all } m} C^{[1^6]1}_{\nu m_\nu,\tilde{\nu} m_{\tilde{\nu}}} C^{\tilde{\nu}_{\alpha'} m_{\tilde{\nu}}}_{\mu' m_{\mu'},f m_f} C^{\mu'_{\beta'} m_{\mu'}}_{\sigma m_c,\sigma_J m_J} \left| \begin{matrix}[\nu]\\ m_\nu W_\nu\end{matrix} \right\rangle \left| \begin{matrix}[f]\\ m_f W_f\end{matrix} \right\rangle \left| \begin{matrix}[\sigma]\\ m_c W_c\end{matrix} \right\rangle \left| \begin{matrix}[\sigma_J]\\ m_J W_J\end{matrix} \right\rangle . \tag{8}$$

The multiquark wave functions in different schemes are related by a unitary transformation, so we have

$$\Phi^{CS}_i=\sum_j C_{ij}\Phi^{FS}_j. \tag{9}$$

where i,j represent intermediate quantum numbers $\tilde{\nu}_{\alpha'},\mu'_{\beta'},\tilde{\nu}_\alpha,\mu_\beta$. Substituting Eqs.(7) and (8) into Eq.(9) and making use of the orthonormality of bases and unitary conditions of CG coefficients, we directly derive the expression for the transformation coefficients, C_{ij}:

$$C_{ij}=\sum_{m_c,m_J,m_\mu,m_f} C^{\tilde{\nu}_{\alpha'} m_{\tilde{\nu}}}_{\mu' m_{\mu'},f m_f} C^{\mu'_{\beta'} m_{\mu'}}_{\sigma m_c,\sigma_J m_J} C^{\tilde{\nu}_\alpha m_{\tilde{\nu}}}_{\sigma m_c,\mu m_\mu} C^{\mu_\beta m_\mu}_{f m_f,\sigma_J m_J}. \tag{10}$$

Comparing Eq.(10) with the definition of the Racah coefficients of the permutation group, we see that the C_{ij} are just the Racah coefficients except for a phase factor,

$$C_{ij}=U(\sigma\sigma_J\tilde{\nu}f;\mu'\mu)^{\beta'\alpha'}_{\beta\alpha}\epsilon(\sigma_J f\mu_\beta) \tag{11}$$

where α, α', β, and β' are multiplicity indices appearing in the coupling, $\sigma\times\mu\to\tilde{\nu}$, $\mu'\times f\to\tilde{\nu}$, $f\times\sigma_J\to\mu$, $\sigma\times\sigma_J\to\mu'$, respectively, and the phase factor $\epsilon(\sigma_J f\mu_\beta)$ comes from the interchange of the order of f and σ_J in the coupling $f\times\sigma_J\to\mu_\beta$.

In order to obtain the *rep* transformation coefficients, the Racah coefficients of the permutation group are needed. (The phase factor $\epsilon(\sigma_J f\mu_\beta)$ is already

available[5]). We are unaware of any publication of the complete Racah coefficients for the permutation group (up to S(6)). Vanagas has given algebraic expressions for some simple cases[6]. Because of the phase choices available for the SU(n) CG coefficients, one has to choose a systematic and phase consistent method to calculate the Racah coefficients, especially for physical applications.

We use the genealogical method to calculate the Racah coefficients of S(n) ($n \leq 6$) based on the existing phase consistent isoscalar factors (ISF) of S(n) $\supset$ S($n-1$) ($n \leq 6$). We start from the trivial S(2) Racah coefficients, by using the formula[7]

$$U(\nu_1\nu_2\nu\nu_3;\nu_{12}\nu_{23})^{\beta_{12}\beta}_{\beta_{23}\beta'} = \sum_{\nu'_1\nu'_2\nu'_3\nu'_{12}\nu'_{23}\theta_{12}\theta_{23}\theta\theta'}^{\text{fixed } \nu'} U(\nu'_1\nu'_2\nu'\nu'_3;\nu'_{12}\nu'_{23})^{\theta_{12}\theta}_{\theta_{23}\theta'} \\ C^{\nu_{12}\beta_{12},\nu'_{12}\theta_{12}}_{\nu_1\nu'_1,\nu_2\nu'_2} C^{\nu\beta,\nu'\theta}_{\nu_{12}\nu'_{12},\nu_3\nu'_3} C^{\nu_{23}\beta_{23},\nu'_{23}\theta_{23}}_{\nu_2\nu'_2,\nu_3\nu'_3} C^{\nu\beta',\nu'\theta'}_{\nu_1\nu'_1,\nu_{23}\nu'_{23}} \qquad (12)$$

where ν's and β's are Yamanouchi labels and multiplicity indices of S(n), and ν''s and θ's are those of S($n-1$). $C^{\nu_{12}\beta_{12},\nu'_{12}\theta_{12}}_{\nu_1\nu'_1,\nu_2\nu'_2}, \cdots$ are the S(n) $\supset$ S($n-1$) isoscalar factors. The Racah coefficients of S(n) can be obtained sequentially in n.

III. Results

We have carried out the calculation of the Racah coefficients of S(n) for $n \leq 6$, based on this genealogical method. The Racah coefficients of S(6) needed in the transformation between the FS scheme and the CS scheme are given in Table 1. The *rep* transformation coefficients between the FS and CS schemes in the u, d, and s 3-flavor world are obtained with the aid of parts of the table of phase factors given by Chen *et al*[5], which are shown in Table 2.

We reproduce Tables VII and VIII of Stancu[4] for the transformation between the FS scheme and the CS scheme with the exception of the absolute phase of

some bases. (For (YIJ)=(201), the phases of [42][3111] and [42][2111] for the CS scheme should be negative. For (YIJ)=(200), no phases need be changed). We also expand the physical bases in terms of symmetry bases in the CS scheme. Tables 3–5 show the results for (YIJ)=(210), (201) and (000), respectively.

Here, we want to emphasize that the importance of phase consistency. Our result for (YIJ)=(201) is different from that of Table IX of Stancu. This is due to a phase inconsistency in the calculation of Stancu, as we now show. In expressing the physical bases in terms of symmetry bases in the CS scheme through the expression of physical bases in the FS scheme and the transformation between the FS scheme and CS scheme, we need two successive transformations. In either step, the absolute phase choice of each basis can be chosen arbitrarily, but one has to use the same phase for each of the intermediate FS coupling symmetry bases to calculate the *rep* transformation coefficients in either step.

Since we can reproduce Tables VII and VIII of Stancu, the phase choice for the FS coupling symmetry bases of Stancu is the same as ours. However the phase choice for the FS coupling symmetry bases of Harvey is different[3]. Stancu directly combined Harvey's transformation Table XI with her Table VII to obtain her Table IX. The phase inconsistency in making these two transformations produces the incorrect entries in Table IX. This can be seen by the following general argument.

For the nucleon, N, the CS symmetry must be $[\mu_i]$=[21] due to the color singlet $[1^3]$ and 1/2 spin [21] symmetry properties. Similarly, for the NN physical basis, the possible CS symmetries, according to the Littlewood rule, are [42],[411],[33], [321],[2211] and [3111]. The symmetry [21111] cannot appear in the coupling of [21]×[21], and so the entry in the last column of the first row in the Table IX of Stancu should be zero. Similar arguments can be applied to the 2nd, 3rd and 4th

columns of the 2nd row in the same Table; all these entries should be zero.

In conclusion, we emphasize again that one must be extremely careful in quoting results from different papers because of the lack of unified phase conventions.

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Table. 1.

The Racah coefficients $U(\nu_1\nu_2\nu\nu_3;\nu_{12}\nu_{23})^{\beta_{12}\beta}_{\beta_{23}\beta'}$ of S(6) with $[\nu_1]$=[222], $\beta_{12}=1$ and $\beta'=1$. The meaning of the table heading is as follows:

$[\nu]$		
$[\nu_2];[\nu_3]$		$[\nu_{23}]_{\beta_{23}}$
	$[\nu_{12}]_{\beta}$	

where the symmetry originally denoted by Young diagrams is denoted by numbers defined in table 6. All the entries represent the square of the Racah coefficients and a minus sign signifies a negative coefficient value (This applies to all the Tables of this paper).

$[\nu]=[1^6]$

1;5		5
	7	1

2;3		5
	9	1

2;6		5
	6	1

3;2		5
	10	1

3;4		5
	8	-1

3;5		5
	7	-1

3;6		5
	6	-1

3;9		5
	3	-1

5;1		5
	11	1

5;3		5
	9	-1

5;7		5
	5	-1

5;8		5
	4	-1

$[\nu]=[21^5]$

1;3		3
	7	1

1;6		6
	7	1

2;2		3
	9	1

2;3	3	6
6	16/25	-9/25
9	9/25	16/25

2;4	3	6
6	-24/25	-1/25
9	1/25	-24/25

2;5		6	3
	6	1	0
	9	0	-1

2;6		3	6_1	6_2
	6_1	-18/25	-1/25	6/25
	6_2	-3/25	-6/25	-16/25
	9	4/25	-18/25	3/25

2;7		6
	6	-1

2;8		6
	6	1

2;9		6
	6	1

3;1		3
	10	1

3;2		3	6
	8	-4/5	1/5
	10	1/5	4/5

3;3		3_1	3_2	6_1	6_2
	6	-16/45	-16/45	1/90	5/18
	7	1/9	-4/9	2/9	-2/9
	8	-4/45	-4/45	-49/90	-5/18
	10	4/9	-1/9	-2/9	2/9

3;4		3	6_1	6_2
	6	-8/15	-2/15	-1/3
	8	4/15	1/15	-2/3
	10	-1/5	4/5	0

3;5		6
	6	-1

3;6		3_1	3_2	6_1	6_2	6_3
	3	9/40	9/40	0	1/20	1/2
	6_1	-1/5	1/5	4/25	-2/5	1/25
	6_2	-3/10	-1/30	-32/75	0	6/25
	7	1/8	1/8	-2/5	-1/4	-1/10
	8	-3/20	5/12	-1/75	3/10	-3/25

3;7		3	6
	3	9/25	-16/25
	6	16/25	9/25

3;8		3	6_1	6_2
	3	-18/25	1/5	2/25
	6	-16/75	-2/15	-49/75
	8	-1/15	-2/3	4/15

3;9		6_1	6_2
	3	1/10	-9/10
	6	9/10	1/10

3;10		6
	3	-1

5;2		3	6
	9	-16/25	9/25
	11	9/25	16/25

5;3		6
	9	1

5;4		3	6
	4	3/5	2/5
	9	-2/5	3/5

5;5		3
	9	1

5;6		3	6_1	6_2
	4	3/10	1/10	3/5
	5	-9/20	-3/20	2/5
	9	-1/4	3/4	0

5;8		6
	4	1

5;9		3	6
	4	4/5	-1/5
	5	-1/5	-4/5

5;10		6
	4	-1

$[\nu] = [2211]$

1;2		2
	7	1

1;4		4
	7	1

1;5		5
	7	1

1;6		6
	7	1

1;9		9
	7	1

2;1		2
	9	1

2;2		2	4
	6	-4/5	1/5
	9	1/5	4/5

2;3		2	4	5	6
	6_1	-2/9	-49/90	-2/9	-1/90
	6_2	2/9	-5/18	2/9	-5/18
	9_1	-4/9	4/45	1/9	-16/45
	9_2	-1/9	-4/45	4/9	16/45

2;4		2	4	6
	6_1	4/5	1/15	2/15
	6_2	0	2/3	-1/3
	9	-1/5	4/15	8/15

2;6		4	5	6_1	6_2	9
	6_1	1/75	-2/5	-4/25	-32/75	0
	6_2	3/10	1/4	-2/5	0	-1/20
	6_3	-3/25	1/10	1/25	-6/25	-1/2
	9_1	3/20	1/8	1/5	-3/10	9/40
	9_2	-5/12	1/8	-1/5	-1/30	9/40

2;5		6
	6	1

2;7		6	9
	6	9/25	16/25
	9	16/25	-9/25

2;8		4	6	9
	6_1	2/3	-2/15	1/5
	6_2	4/15	49/75	-2/25
	9	-1/15	16/75	18/25

2;9		6	9
	6_1	9/10	-1/10
	6_2	-1/10	-9/10

2;10		9
	6	-1

3;2		2	4	5	6
	6	20/81	-25/81	20/81	-16/81
	7	-16/81	20/81	25/81	-20/81
	8	20/81	16/81	20/81	25/81
	10	25/81	20/81	-16/81	-20/81

3;3		2	4	6_1	6_2
	6_1	2/9	1/18	25/36	1/36
	6_2	-2/9	1/2	1/36	-1/4
	8	4/9	0	-1/18	-1/2
	10	1/9	4/9	-2/9	2/9

3;4		2	4_1	4_2	5	6_1	6_2	9
	3	2/5	1/16	5/16	1/16	0	4/25	1/400
	6_1	4/81	5/18	1/162	-40/81	4/81	-10/81	0
	6_2	128/405	-1/9	-5/81	4/81	10/81	-361/2025	-4/25
	7	10/81	1/16	-245/1296	25/1296	-40/81	-4/81	1/16
	8_1	0	25/144	-5/16	1/16	5/18	1/9	1/16
	8_2	8/81	-5/16	-25/1296	-245/1296	1/162	5/81	5/16
	10	1/81	0	-8/81	-10/81	-4/81	128/405	-2/5

3;5		2	4	5	6	9
	3	1/5	1/40	5/16	-4/25	-121/400
	6	-64/405	-8/405	-20/81	-841/2025	-4/25
	7	25/81	25/648	-121/1296	-20/81	5/16
	8	-2/81	289/324	-25/648	8/405	-1/40
	10	-25/81	2/81	25/81	-64/405	1/5

3;6		2	4_1	4_2	5	6_1	6_2	6_3	9_1	9_2
	3_1	1/8	1/8	1/5	-1/8	9/125	-49/200	-49/500	-1/100	0
	3_2	1/8	-1/8	0	-1/8	1/5	1/8	1/20	0	1/4
	6_1	8/45	0	49/225	2/45	-16/5625	32/1125	1444/5625	9/125	-1/5
	6_2	-16/81	-1/81	289/810	-1/324	-32/1125	16/2025	-242/10125	49/200	1/8
	6_3	8/405	10/81	-169/2025	-169/810	-1444/5625	-242/10125	6889/50625	49/500	1/20
	7	25/324	-25/81	-5/162	-25/324	-2/45	-1/324	-169/810	1/8	-1/8
	8_1	-16/81	-25/324	5/162	-25/81	0	-1/81	10/81	-1/8	-1/8
	8_2	5/162	5/162	4/81	-5/162	-49/225	289/810	-169/2025	-1/5	0
	10	-4/81	16/81	-5/162	-25/324	8/45	16/81	-8/405	1/8	-1/8

3;7		6
	6	-1

3;8		4	6_1	6_2	9
	3	-2/5	-1/25	-2/5	-4/25
	6_1	-2/45	196/225	-2/45	1/25
	6_2	4/9	2/45	-1/9	-2/5
	8	-1/9	2/45	4/9	-2/5

3;9		4	5	6_1	6_2	9_1	9_2
	3_1	9/40	1/16	-8/25	0	-121/400	-9/100
	3_2	-1/10	1/4	1/50	1/2	-9/100	1/25
	6_1	16/45	2/9	49/900	1/36	8/25	-1/50
	6_2	0	2/9	-1/36	-1/4	0	1/2
	7	5/72	-25/144	-2/9	2/9	1/16	1/4
	8	1/4	-5/72	16/45	0	-9/40	1/10

3;10		6	9
	3	9/25	16/25
	6	-16/25	9/25

3;11		9
	3	-1

5;1		5
	9	1

5;2		6
	9	1

5;3		2	4	5	6	9
	4	2/9	1/36	25/72	-8/45	-9/40
	5	-1/9	25/72	-25/144	-16/45	-1/80
	9_1	1/9	-25/72	-49/144	-4/45	-9/80
	9_2	4/9	1/18	-1/36	-1/45	9/20
	11	1/9	2/9	-1/9	16/45	-1/5

5;4		4	6
	4	1/3	-2/3
	9	-2/3	-1/3

5;6		2	4	6_1	6_2	9
	4_1	0	-5/12	1/3	0	-1/4
	4_2	1/2	1/6	1/30	-1/5	-1/10
	5	1/4	0	3/20	2/5	1/5
	9_1	1/8	-3/8	-3/40	-1/5	9/40
	9_2	-1/8	1/24	49/120	-1/5	9/40

5;7		4	5	9
	4	1/4	-5/8	-1/8
	5	-5/8	-1/16	-5/16
	9	-1/8	-5/16	9/16

5;8		4	5	6	9
	4_1	-5/24	1/16	1/6	-9/16
	4_2	3/8	5/16	3/10	1/80
	5	3/8	-5/16	0	-5/16
	9	-1/24	-5/16	8/15	9/80

5;9		6
	4	-1

5;10		6	9
	4	-1/5	-4/5
	5	-4/5	1/5

$[\nu] = [222]$

1;1		1
	7	1

1;3		3
	7	1

1;7		7
	7	1

1;8		8
	7	1

2;2		1	3
	6	16/25	9/25
	9	9/25	-16/25

2;3		3
	6	1

2;4		3	8
	6	-3/5	2/5
	9	2/5	3/5

2;5		3
	9	1

2;6		3	7	8
	6_1	3/4	-3/20	-1/10
	6_2	0	2/5	-3/5
	9	-1/4	-9/20	-3/10

2;8		8
	6	-1

2;9		7	8
	6	4/5	1/5
	9	1/5	-4/5

2;10		8
	6	1

3;1		3
	7	1

3;2		3
	6	1

3;3		1	3_1	3_2	7	8
	3	1/5	9/80	9/20	1/80	-9/40
	6	16/45	-4/45	1/45	-16/45	8/45
	7	1/9	49/144	-1/36	25/144	25/72
	8	2/9	-25/72	-1/18	25/72	-1/36
	10	1/9	1/9	-4/9	-1/9	-2/9

3;4		3	8
	6	-1/3	-2/3
	8	-2/3	1/3

3;6		3_1	3_2	7	8_1	8_2
	3	-9/40	-9/40	-1/5	-1/4	-1/10
	6_1	3/40	-49/120	-3/20	1/3	1/30
	6_2	1/5	1/5	-2/5	0	-1/5
	8	3/8	-1/24	0	-5/12	1/6
	10	1/8	-1/8	1/4	0	-1/2

3;7		3	7	8
	3	9/16	5/16	-1/8
	7	5/16	-1/16	5/8
	8	-1/8	5/8	1/4

3;8		3	7	8_1	8_2
	3	9/80	5/16	-9/16	-1/80
	6	8/15	0	1/6	-3/10
	7	-5/16	5/16	1/16	-5/16
	8	1/24	3/8	5/24	3/8

3;9		8
	6	1

3;10		7	8
	3	-1/5	4/5
	6	-4/5	-1/5

5;2		3
	9	1

5;4		3	7	8
	4	-3/8	-3/8	-1/4
	5	9/16	-1/16	-3/8
	9	1/16	-9/16	3/8

5;5		1	3	7	8
	4	2/5	9/40	-1/8	-1/4
	5	1/5	9/80	9/16	1/8
	9	9/25	-121/400	-9/80	9/40
	11	1/25	-9/25	1/5	-2/5

5;6		3	8
	4	-3/4	1/4
	9	-1/4	-3/4

5;8		8
	4	-1

5;9		3	7	8
	4	1/8	5/8	-1/4
	5	-5/16	-1/16	-5/8
	9	9/16	-5/16	-1/8

5;11		7
	5	1

Table 2. The triplets $(\sigma_J f \mu_\beta)$ which have $\epsilon(\sigma_J f \mu_\beta) = -1$.

$[\sigma_J];[f]$	$[\mu]_\beta$	$[\sigma_J];[f]$	$[\mu]_\beta$	$[\sigma_J];[f]$	$[\mu]_\beta$	$[\sigma_J];[f]$	$[\mu]_\beta$	$[\sigma_J];[f]$	$[\mu]_\beta$
2;2	4	3;3	4	3;6	7	3;9	9_2	5;6	10
2;3	6	3;3	6_2	3;6	9_1	3;9	10	5;7	11
2;6	8	3;3	8	3;6	9_2	3;9	11		
2;6	7	3;4	8	3;6	10	5;5	8		
2;8	10	3;5	9	3;7	10	5;6	8		

$\epsilon(\sigma_J f \mu_\beta) = \epsilon(f \sigma_J \mu_\beta)$. The cases $\epsilon(\sigma_J f \mu_\beta) = 1$ which occur when σ_J or f are either totally symmetric or totally antisymmetric are not shown.

All the triplets needed in this calculation, which are not listed in Table 2, have $\epsilon(\sigma_J f \mu_\beta) = 1$.

Table 3. The transformation between physical and symmetry bases in the CS scheme for (YIJ)=(210). The labels of the columns are the values of $[\nu]$ $[\mu]_\beta$ $[f]$.

$B_1 B_2$	1 9 3	3 5 3	3 4 3	3 9_1 3	3 9_2 3	3 11 3
NN	$-\sqrt{1/9}$	$\sqrt{1/2}$	$-\sqrt{1/4}$	$-\sqrt{1/36}$	$\sqrt{1/9}$	0
$\Delta\Delta$	$\sqrt{4/45}$	0	0	$\sqrt{16/45}$	$\sqrt{16/45}$	$\sqrt{1/5}$

Table 4. Same as Table 3, but for (YIJ)=(201).

$B_1 B_2$	1 8 5	3 3 5	3 6 5	3 8 5	3 7 5	3 10 5
NN	$-\sqrt{1/9}$	$\sqrt{9/20}$	$-\sqrt{16/45}$	$\sqrt{1/36}$	$-\sqrt{1/18}$	0
$\Delta\Delta$	$\sqrt{4/45}$	0	0	$\sqrt{16/45}$	0	$-\sqrt{5/9}$

Table 5. Same as Table 3, but for (YIJ)=(000).

B_1B_2	1 9 3	1 4 8	2 5 6	2 4 6	2 9 6	3 5 3	3 4 3
$\overline{N\Xi}$	0	0	$\sqrt{1/6}$	$-\sqrt{1/4}$	$-\sqrt{5/36}$	0	0
$\Sigma\Sigma$	$\sqrt{1/360}$	$-\sqrt{3/40}$	0	0	0	$-\sqrt{1/80}$	$\sqrt{1/160}$
$\widetilde{N\Xi}$	$\sqrt{1/30}$	$\sqrt{1/10}$	0	0	0	$-\sqrt{3/20}$	$\sqrt{3/40}$
$\Lambda\Lambda$	$-\sqrt{3/40}$	$\sqrt{1/40}$	0	0	0	$\sqrt{27/80}$	$-\sqrt{27/160}$
$\Sigma^*\Sigma^*$	$\sqrt{4/45}$	0	0	0	0	0	0
	3 9_1 3	3 9_2 3	3 11 3	3 5_2 6	3 4 6	3 9_1 6	3 9_2 6
$\overline{N\Xi}$	0	0	0	0	0	0	0
$\Sigma\Sigma$	$\sqrt{1/1440}$	$-\sqrt{1/360}$	0	$\sqrt{1/10}$	$\sqrt{3/20}$	$\sqrt{3/40}$	$-\sqrt{3/40}$
$\widetilde{N\Xi}$	$\sqrt{1/120}$	$-\sqrt{1/30}$	0	$\sqrt{1/10}$	$\sqrt{1/20}$	$\sqrt{1/40}$	$-\sqrt{1/40}$
$\Lambda\Lambda$	$-\sqrt{3/160}$	$\sqrt{3/40}$	0	$\sqrt{1/10}$	$\sqrt{1/20}$	$\sqrt{1/40}$	$-\sqrt{1/40}$
$\Sigma^*\Sigma^*$	$\sqrt{16/45}$	$\sqrt{16/45}$	$\sqrt{1/5}$	0	0	0	0
	3 5 8	3 4 8	3 9 8	4 5 6	4 9 6		
$\overline{N\Xi}$	0	0	0	$-\sqrt{1/3}$	$-\sqrt{1/9}$		
$\Sigma\Sigma$	$-\sqrt{3/16}$	$-\sqrt{3/160}$	$-\sqrt{3/32}$	0	0		
$\widetilde{N\Xi}$	$\sqrt{1/4}$	$\sqrt{1/40}$	$\sqrt{1/8}$	0	0		
$\Lambda\Lambda$	$\sqrt{1/16}$	$\sqrt{1/160}$	$\sqrt{1/32}$	0	0		
$\Sigma^*\Sigma^*$	0	0	0	0	0		

Table 6. Column label coding, used in Table 1-5, for the Young diagrams.

1	2	3	4	5	6	7	8	9	10	11
[6]	[51]	[42]	[411]	[33]	[321]	[222]	[3111]	[2211]	[211111]	[111111]